Problem Set 8 due November 11, at 10 AM, on Gradescope (via Stellar)

Please list all of your sources: collaborators, written materials (other than our textbook and lecture notes) and online materials (other than Gilbert Strang's videos on OCW).

Give complete solutions, providing justifications for every step of the argument. Points will be deducted for insufficient explanation or answers that come out of the blue

Problem 1: Consider the matrices:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} x & y \\ z & t \end{bmatrix}$$

(1) Prove that $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$ by computing them explicitly. *(10 points)*

Proof. Explicitly:

$$AB = \begin{bmatrix} ax+bz & ay+bt \\ cx+dz & cy+dt \end{bmatrix} \quad \Rightarrow \quad \mathrm{Tr}(AB) = ax+bz+cy+dt$$

$$BA = \begin{bmatrix} xa+yc & xb+yd \\ za+tc & zb+td \end{bmatrix} \quad \Rightarrow \quad \mathrm{Tr}(BA) = xa+zb+yc+td$$

□

(2) Use the previous part to prove that $\mathrm{Tr}(D) = \mathrm{Tr}(VDV^{-1})$ for any square matrix D and any invertible matrix V. *(5 points)*

Proof. Just apply the identity $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$ for $A = DV^{-1}$ and $B = V$.

□

Problem 2: Consider any square matrices A, B, V with the latter being invertible, such that:

$$B = VAV^{-1}$$

(1) Prove that $B - \lambda I = V(A - \lambda I)V^{-1}$. *(5 points)*

Proof. We have:

$$V(A-\lambda I)V^{-1} = VAV^{-1} - \lambda VIV^{-1} = VAV^{-1} - \lambda I = B - \lambda I$$

□

(2) Use the previous part to prove that:

$$\det(B - \lambda I) = \det(A - \lambda I)$$

(5 points)

Proof. We have:

$$\det(B - \lambda I) = \det V(A - \lambda I)V^{-1} = \det V \cdot \det(A - \lambda I) \cdot \det V^{-1} = \det(A - \lambda I)$$

since $\det V^{-1} = \frac{1}{\det V}$.

□

(3) Use the previous part to prove that the conjugate matrices A and B have the same eigenvalues. *(5 points)*

Proof. The previous part says that A and B have the same characterstic polynomial, hence they have the same eigenvalues.

□

(4) Use the previous part to give another proof of problem 1.2. *(5 points)*

Proof. Since the trace of a matrix is just the sum of the eigenvalues, part (3) implies that:

$$\mathrm{Tr}(A) = \mathrm{Tr}(VAV^{-1})$$

which is just what problem 1.2 states.

□

Problem 3: Let A be an $m \times n$ matrix and B be an $n \times m$ matrix. Prove that the any <u>non-zero</u> eigenvalue of the square matrix AB is also an eigenvalue of the matrix BA. *(10 points)*

Proof. Assume $\lambda \neq 0$ is an eigenvalue of AB. This means that:

$$AB\boldsymbol{v} = \lambda \boldsymbol{v} \tag{1}$$

for some non-zero vector $\boldsymbol{v}$. Multiplying the relation above on the left with B gives:

$$BAB\boldsymbol{v} = \lambda B\boldsymbol{v}$$

If we let $\boldsymbol{w} = B\boldsymbol{v}$, then we get:

$$BA\boldsymbol{w} = \lambda \boldsymbol{w}$$

This implies that λ is an eigenvalue of BA, because $\boldsymbol{w} \neq 0$ (otherwise, the fact that $\boldsymbol{w} = B\boldsymbol{v} = 0$ would force $\lambda = 1$ in (1)).

□

Problem 4: Consider the matrix:

$$A = \begin{bmatrix} -4 & -3 & -2 \\ 3 & 3 & 1 \\ 15 & 8 & 7 \end{bmatrix}$$

(1) Find an eigenvalue λ of A, and compute an eigenvector $\boldsymbol{v}$. *(10 points)*

Proof. The characteristic polynomial is:

$$p(\lambda) = \det\left(\begin{bmatrix} -4-\lambda & -3 & -2 \\ 3 & 3-\lambda & 1 \\ 15 & 8 & 7-\lambda \end{bmatrix}\right)$$

We may compute this by cofactor expansion along the second row, and obtain:

$$p(\lambda) = -3\cdot\det\left(\begin{bmatrix} -3 & -2 \\ 8 & 7-\lambda \end{bmatrix}\right) + (3-\lambda)\cdot\det\left(\begin{bmatrix} -4-\lambda & -2 \\ 15 & 7-\lambda \end{bmatrix}\right) - 1\cdot\det\left(\begin{bmatrix} -4-\lambda & -3 \\ 15 & 8 \end{bmatrix}\right) =$$

$$= -3(3\lambda-5) + (3-\lambda)(\lambda^2-3\lambda+2) - (-8\lambda+13) = -\lambda^3+6\lambda^2-12\lambda+8 = (2-\lambda^3)$$

So the only eigenvalue is $\lambda = 2$ (with algebraic multiplicity 3). To compute an eigenvector, we must find:

$$\boldsymbol{v} \in N(A-2I) = N\left(\begin{bmatrix} -6 & -3 & -2 \\ 3 & 1 & 1 \\ 15 & 8 & 5 \end{bmatrix}\right) \overset{\text{RREF}}{\rightsquigarrow} N\left(\begin{bmatrix} 1 & 0 & \frac{1}{3} \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right)$$

So if $\boldsymbol{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, then we require:

$$0 = \begin{bmatrix} 1 & 0 & \frac{1}{3} \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x+\frac{z}{3} \\ y \\ 0 \end{bmatrix}$$

Therefore, all eigenvectors are constant multiples of:

$$\boldsymbol{v} = \begin{bmatrix} -\frac{1}{3} \\ 0 \\ 1 \end{bmatrix}$$

so the geometric multiplicity of the eigenvalue $\lambda = 2$ is 1.

$\square$

(2) Compute a vector $\boldsymbol{w}$ such that $(A-\lambda I)\boldsymbol{w} = \boldsymbol{v}$ and a vector $\boldsymbol{z}$ such that $(A-\lambda I)\boldsymbol{z} = \boldsymbol{w}$. *(10 points)*

Proof. Let $\boldsymbol{w} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ so we are solving for:

$$\begin{bmatrix} -6 & -3 & -2 \\ 3 & 1 & 1 \\ 15 & 8 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -\frac{1}{3} \\ 0 \\ 1 \end{bmatrix}$$

This is also done by Gauss-Jordan elimination on the system above (which is done similar to part (1)):

$$\begin{bmatrix} 1 & 0 & \frac{1}{3} \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -\frac{1}{9} \\ \frac{1}{3} \\ 0 \end{bmatrix} \qquad \Rightarrow \qquad \begin{cases} x + \frac{z}{3} = -\frac{1}{9} \\ y = \frac{1}{3} \end{cases}$$

so a solution is $\boldsymbol{w} = \begin{bmatrix} -\frac{1}{9} \\ \frac{1}{3} \\ 0 \end{bmatrix}$. *Note: you could have given z any other value, if you solved correctly for x and y from the system above, you would have gotten another valid formula for $\boldsymbol{w}$.*

Now let $\boldsymbol{z} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ so we are solving for:

$$\begin{bmatrix} -6 & -3 & -2 \\ 3 & 1 & 1 \\ 15 & 8 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -\frac{1}{9} \\ \frac{1}{3} \\ 0 \end{bmatrix}$$

This is also done by Gauss-Jordan elimination on the system above (which is done similar to part (1)):

$$\begin{bmatrix} 1 & 0 & \frac{1}{3} \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{8}{27} \\ -\frac{5}{9} \\ 0 \end{bmatrix} \qquad \Rightarrow \qquad \begin{cases} x + \frac{z}{3} = \frac{8}{27} \\ y = -\frac{5}{9} \end{cases}$$

so a solution is $\boldsymbol{z} = \begin{bmatrix} \frac{8}{27} \\ -\frac{5}{9} \\ 0 \end{bmatrix}$. *Note: you could have given z any other value, if you solved correctly for x and y from the system above, you would have gotten another valid formula for $\boldsymbol{z}$.*

$\square$

(3) Consider the matrix $V = \left[\boldsymbol{v}|\boldsymbol{w}|\boldsymbol{z}\right]$ and compute:

$$V^{-1}AV$$

Congratulations: you just computed the Jordan normal form of A. *(10 points)*

Proof. By the computations in parts (1) and (2), we have:

$$V = \begin{bmatrix} -\frac{1}{3} & -\frac{1}{9} & \frac{8}{27} \\ 0 & \frac{1}{3} & -\frac{5}{9} \\ 1 & 0 & 0 \end{bmatrix}$$

Note: this formula for V depends on the choice you made for $\boldsymbol{w}$ and $\boldsymbol{z}$; other choices would lead to other versions of V, all equally good The inverse of this matrix is quite easy to compute (say by the cofactor formula for the inverse):

$$V^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 15 & 8 & 5 \\ 9 & 3 & 3 \end{bmatrix}$$

Thus we find:

$$V^{-1}AV = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

which is the Jordan normal form of A.

□

Problem 5: Consider the matrix:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

(1) Compute $A^2, A^3, A^4, A^5, \ldots$ and write a formula for the matrix exponential e^{At}. *(10 points)*

Proof. We have:

$$A^2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \text{and} \qquad A^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Since the zero matrix multiplied with any other matrix is 0, this implies $A^4 = A^5 = \ldots = 0$. Therefore, the Taylor series for the matrix exponential reads:

$$e^{At} = I + At + \frac{At^2}{2} = \begin{bmatrix} 1 & t & \frac{t^2}{2} \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}$$

□

(2) Use formula (205) to find the solution of the system $\dot{\boldsymbol{u}}(t) = A\boldsymbol{u}(t)$. *(10 points)*

Proof. The solution is:

$$\boldsymbol{u}(t) = e^{At}\boldsymbol{a} = \begin{bmatrix} 1 & t & \frac{t^2}{2} \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} a_1 + ta^2 + \frac{t^2}{2}a_3 \\ a_2 + ta_3 \\ a_3 \end{bmatrix} \tag{2}$$

for arbitrary constants a_1, a_2, a_3.

□

(3) Find the complete solution to the differential equation $f'''(t) = 0$ by setting it up as the system of differential equations in part (2). *(5 points)*

Proof. Let:

$$\boldsymbol{u}(t) = \begin{bmatrix} f(t) \\ f'(t) \\ f''(t) \end{bmatrix}$$

Then the equation $f'''(t) = 0$ is precisely $\dot{\boldsymbol{u}}(t) = A\boldsymbol{u}(t)$. So from part (2), we obtain:

$$f(t) = a_1 + ta^2 + \frac{t^2}{2}a_3$$

for arbitrary constants a_1, a_2, a_3. $\square$